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Convergence rate of Fourier–Laplace series of L^2-functions

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Abstract

The almost everywhere convergence rates of Fourier–Laplace series are given for functions in certain subclasses of $L^2(\Sigma_{n-1})$ defined in terms of moduli of continuity.
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1. Introduction and main results

Let $n \geqslant 3$ and let $\Sigma_{n-1} = \{(x_1, \dots, x_n) : x_1^2 + \cdots + x_n^2 = 1\}$ be the unit sphere of $\mathbb{R}^n$ equipped with the normal Lebesgue measure. Let $f \in L^2(\Sigma_{n-1})$ and let

$$f \sim \sigma(f)(x) := \sum_{k=0}^{\infty} Y_k(f)(x) \tag{1}$$

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be the Fourier–Laplace series of f, where Y_k is the projection operator from $L^2(\Sigma_{n-1})$ to the space of all spherical harmonics of degree k. The Cesàro means of order δ of $\sigma(f)$ are defined as usual by

$$\sigma_N^\delta(f) := (A_N^\delta)^{-1} \sum_{k=0}^{N} A_{N-k}^\delta Y_k(f),$$

where

$$A_N^\delta = \frac{\Gamma(N+\delta+1)}{\Gamma(\delta+1)\Gamma(N+1)}, \quad \delta > -1, \tag{2}$$

are the coefficients of the power series of the function $(1-x)^{-\delta-1}$, $|x|<1$; i.e.

$$(1-x)^{-\delta-1} = \sum_{k=0}^{\infty} A_k^\delta x^k. \tag{3}$$

It is obvious that $\sigma_N^0(f) = \sum_{k=0}^{N} Y_k(f)$ is just the Nth partial sum of the Fourier–Laplace series of f.

Our main purpose is to find the convergence rate of $\sigma_N^0(f)$ on a set of full measure in Σ_{n-1} for any f in certain subclasses of $L^2(\Sigma_{n-1})$ defined in terms of modulus of continuity.

In order to define modulus of continuity on the sphere, we first introduce the translation operator S_θ with step $\theta \in \mathbb{R}$. As in [7, p. 58], we define

$$S_\theta(f)(x) := \frac{1}{|\Sigma_{n-2}|} \int_{\{y \in \Sigma_{n-1}:\, xy=0\}} f(x\cos\theta + y\sin\theta)\, d\ell(y),$$

here $x \in \Sigma_{n-1}$ and $d\ell(y)$ denotes the usual Lebesgue measure elements on the $n-2$ dimensional manifold $\{y \in \Sigma_{n-1}: xy = 0\}$. We know (see [7, p. 61, (2.4.6)]) that, for every $k = 0, 1, 2, \ldots$,

$$Y_k(S_\theta(f)) = P_k^n(\cos\theta) Y_k(f),$$

where P_k^n is Gegenbauer polynomial defined by

$$P_k^n(t) = \frac{P_k^{\left(\frac{n-3}{2},\frac{n-3}{2}\right)}(t)}{P_k^{\left(\frac{n-3}{2},\frac{n-3}{2}\right)}(t)}, \quad |t| \leqslant 1, \tag{4}$$

with $P_k^{(\alpha,\beta)}$ being Jacobi polynomial. By the formulas [6, p. 58, (4.1.1)] and [p. 168, (7.32.2)], we have

$$|P_k^n(\cos\theta)| \leqslant \begin{cases} 1, \\ \dfrac{\gamma}{(k\theta(\pi-\theta))^{\frac{n-2}{2}}}, \end{cases} \quad \text{for all } \theta \in (0, \pi), \tag{5}$$

where $\gamma > 1$ is a constant depending only on n. Then we get

$$||S_\theta(f)||_2 = \left(\sum_{k=0}^{\infty} |P_k^n(\cos\theta)|^2 ||Y_k(f)||_2^2 \right)^{\frac{1}{2}} \leqslant ||f||_2.$$

Thus, we conclude that as an operator from $L^2(\Sigma_{n-1})$ to $L^2(\Sigma_{n-1})$, S_θ has norm 1; that is, $||S_\theta||_{(L^2(\Sigma_{n-1}),L^2(\Sigma_{n-1}))} = 1$. Let I be the identity operator and let s be a positive number. We set $\psi_s(u) \coloneqq (1-u)^{\frac{s}{2}}$. Following [5], we call the operator

$$\Delta_\theta^s \coloneqq (I - S_\theta)^{\frac{s}{2}} = \sum_{k=0}^{\infty} \frac{\psi_s^{(k)}(0)}{k!} S_\theta^k$$

an sth order difference operator. It is obvious that

$$||\Delta_\theta^s||_{(L^2(\Sigma_{n-1}),L^2(\Sigma_{n-1}))} \leqslant \sum_{k=0}^{\infty} \frac{|\psi_s^{(k)}(0)|}{k!} < \infty.$$

We define (as in [5]) the sth order modulus of continuity of a function $f \in L^2(\Sigma_{n-1})$ by

$$\omega_s(f,t)_2 \coloneqq \sup\{||\Delta_u^s f||_2 : 0 < u \leqslant t\}.$$

Since all of our discussion are in $L^2(\Sigma_{n-1})$, in what follows, we will omit the subscription "2" in the norm and in the moduli.

It is well known (see [5]) that, for $0 < \alpha < \beta$ and $f \in L^2(\Sigma_{n-1})$,

$$\omega_\beta(f,t) \leqslant C(\alpha,\beta)\omega_\alpha(f,t), \tag{6}$$

where $C(\alpha,\beta)$ is a constant depending only on α and β.

Definition 1. Let $s > 0, r \in \mathbb{R}$, and $f \in L^2(\Sigma_{n-1})$. If

$$\int_0^1 \frac{\omega_s(f,t)^2}{t} \log^r\left(\frac{2}{t}\right) dt < \infty,$$

then we say that f satisfies the condition $(\{s,r\})$.

Our first result is the following theorem:

Theorem 1. *Let $r \geqslant 1$. If there exists $s > 0$ such that f satisfies the condition $(\{s,r\})$, then*

$$\sigma_N^0(f)(x) - f(x) = o\left(\frac{1}{\log^{\frac{r-1}{2}} N}\right) \quad \text{as } N \to \infty$$

holds on Σ_{n-1} almost everywhere.

Our second theorem is about the case $0 \leqslant r < 1$.

Theorem 2. *Let* $0 \leqslant r < 1$. *If there exists* $s > 0$ *such that* f *satisfies the condition* $(\{s, r\})$, *then*

$$\lim_{N \to \infty} \log^{\frac{r-1}{2}} N(\sigma_N^0(f)(x) - f(x)) = 0$$

holds on Σ_{n-1} *almost everywhere.*

Remark 1. Since the modulus of continuity is defined in square integrable terms, the convergence rate in L^2-norm can be obtained easily. For example, if f satisfies the condition $(\{2, r\})$, i.e.

$$f_{\frac{r+1}{2}} \sim \sum_{k=0}^{\infty} \log^{\frac{r+1}{2}}(k+2)\, Y_k(f) \in L^2(\Sigma_{n-1}),$$

then we can easily get

$$||\sigma_N^0(f) - f||_2 = \left\{ \sum_{k=N+1}^{\infty} \log^{-(r+1)}(k+2) ||\, Y_k(f_{\frac{r+1}{2}})||_2^2 \right\}^{\frac{1}{2}} = o\left(\frac{1}{\log^{\frac{r+1}{2}}(N+2)} \right).$$

This order $\log^{-\frac{r+1}{2}} N$ is better than the almost everywhere convergence rate $\log^{-\frac{r-1}{2}} N$.

Remark 2. Both Theorems 1 and 2 have the same form. However, when $r \geqslant 1$, we really get the convergence rate like Theorem 1; when $0 \leqslant r < 1$, we cannot conclude whether $\sigma_N^0(f)$ converges almost everywhere but get only the almost everywhere convergence of $\log^{\frac{r-1}{2}} N(\sigma_N^0(f) - f)$.

Before proving the theorems, we will introduce some function classes related to the condition $(\{s, r\})$. Given a function $f \in L^2(\Sigma_{n-1})$ and a positive number r, if

$$\sum_{k=0}^{\infty} \log^{2r}(k+2) ||\, Y_k(f)||_2^2 < \infty,$$

then we say $f \in L_r^2(\Sigma_{n-1})$ and write

$$f_r = \sum_{k=0}^{\infty} \log^r(k+2)\, Y_k(f) \quad \text{in } L^2(\Sigma_{n-1}) \text{ sense.} \tag{7}$$

In Section 2, we give a characterization for the function class $L_r^2(\Sigma_{n-1})$ in terms of the modulus of continuity. In Section 3, we give a domination for maximal partial sum with "log" factor of Fourier–Laplace series which plays the key role for proving the main theorems. In Section 4 we complete the proofs of the theorems.

2. Characterization of the class $L^2_r(\Sigma_{n-1})$

Theorem 3. *Let $r > -1$. The necessary and sufficient condition for $f \in L^2_{\frac{r+1}{2}}(\Sigma_{n-1})$ is that f satisfies $(\{2, r\})$.*

Proof. Assume $f \in L^2_{\frac{r+1}{2}}(\Sigma_{n-1})$ first. Then

$$f_{\frac{r+1}{2}}(x) \sim \sigma(f_{\frac{r+1}{2}})(x) := \sum_{k=0}^{\infty} \log^{\frac{r+1}{2}}(k+2)\, Y_k(f)(x).$$

We have

$$\Delta_t^2(f) \sim \sum_{k=1}^{\infty} (1 - P_k^n(\cos t))\, Y_k(f)$$

and

$$||\Delta_t^2(f)||^2 = \sum_{k=1}^{\infty} (1 - P_k^n(\cos t))^2 ||Y_k(f)||^2. \tag{8}$$

Applying the formula (see [6, p. 81], [7, p. 31])

$$\frac{d}{dt} P_k^n(t) = \frac{k(k+n-2)}{n-1} P_{k-1}^{n+2}(t),$$

we have

$$1 - P_k^n(\cos\theta) = \frac{k(k+n-2)}{n-1} P_{k-1}^{n+2}(\cos\xi)(1-\cos\theta),$$

where $\xi \in (0, \theta)$. Taking (5) into account, we hence have

$$0 \leqslant 1 - P_k^n(\cos\theta) \leqslant (k\theta)^2. \tag{9}$$

It follows from (5) and $\frac{n-2}{2} \geqslant \frac{1}{2}$ that

$$|P_k^n(\cos\theta)| \leqslant \tfrac{1}{2} \quad \text{for } \theta \in (0,1) \text{ and } k\theta \geqslant 4\gamma^2. \tag{10}$$

Write

$$\delta_k(t) = \sup\{|1 - P_k^n(\cos\theta)|^2 : 0 \leqslant \theta \leqslant t\}, \quad t \in (0,1).$$

By (8) we get

$$\int_0^1 \frac{\omega_2(f,t)^2}{t} \log^r\left(\frac{2}{t}\right) dt \leqslant \sum_{k=1}^{\infty} \int_0^1 \frac{\delta_k(t)}{t} \log^r\left(\frac{2}{t}\right) dt ||Y_k(f)||^2.$$

Applying (9) and (10), we have

$$\delta_k(t) \leqslant \begin{cases} 1, & 4\gamma^2 k^{-1} < t < 1, \quad k > 4\gamma^2, \\ (kt)^4, & 0 < t \leqslant 4\gamma^2 k^{-1}, \quad k > 4\gamma^2, \\ (kt)^4 \leqslant B_n t^4, & 0 < t < 1, \quad k \leqslant 4\gamma^2, \end{cases} \tag{11}$$

where B_n is a constant depending only on n. From (11) we derive that

$$\int_0^1 \frac{\delta_k(t)}{t} \log^r\left(\frac{2}{t}\right) dt \leqslant B_n \log^{r+1}(k+2).$$

Since $f \in L^2_{\frac{r+1}{2}}(\Sigma_{n-1})$ as an assumption, we see that f satisfies the condition $(\{2, r\})$.

We now assume that f satisfies the condition $(\{2, r\})$. We start from (8). Then

$$\sum_{k=1}^{\infty} \int_0^1 (1 - P_k^n(\cos t))^2 t^{-1} \log^r\left(\frac{2}{t}\right) dt \|Y_k(f)\|^2$$
$$= \int_0^1 \|\Delta_t^2(f)\|^2 t^{-1} \log^r\left(\frac{2}{t}\right) dt < \infty.$$

Hence

$$\sum_{k>4\gamma^2}^{\infty} \int_{4\gamma^2 k^{-1}}^1 (1 - P_k^n(\cos t))^2 t^{-1} \log^r\left(\frac{2}{t}\right) dt \|Y_k(f)\|^2 < \infty.$$

So, by (10) we obtain

$$\sum_{k>4\gamma^2}^{\infty} \int_{4\gamma^2 k^{-1}}^1 t^{-1} \log^r\left(\frac{2}{t}\right) dt \|Y_k(f)\|^2 < \infty,$$

which implies $f \in L^2_{\frac{r+1}{2}}(\Sigma_{n-1})$. □

3. Domination for Fourier–Laplace partial sums with log factors

From (2) we derive the well-known formula for Cesàro numbers:

$$A_k^{\alpha+\beta} = \sum_{j=0}^{k} A_{k-j}^{\alpha-1} A_j^{\beta}, \quad \alpha > 0, \ \beta > -1.$$

By this formula and (3) we get, for $f \in L^2(\Sigma_{n-1})$,

$$\sigma_N^0(f) = \sum_{k=0}^{N} A_{N-k}^{-\frac{1}{2}} A_k^{-\frac{1}{2}} \sigma_k^{-\frac{1}{2}}(f). \tag{12}$$

Denote by σ_*^α the maximal Cesàro operator of order α; that is,

$$\sigma_*^\alpha(f)(x) := \sup\{|\sigma_k^\alpha(f)(x)| : k \in \mathbb{N}\}, \quad \alpha > -1, \ x \in \Sigma_{n-1}.$$

It is well known (see [1]) that, for $\alpha > 0$, σ_*^α is bounded from $L^2(\Sigma_{n-1})$ to $L^2(\Sigma_{n-1})$.

We introduce an operator on $L^2(\Sigma_{n-1})$ as follows:

Definition 2. For $f \in L^2(\Sigma_{n-1})$, define

$$\delta(f) := \left(\sum_{k=0}^{\infty} |A_k^{-\frac{1}{2}}(\sigma_k^{-\frac{1}{2}}(f) - \sigma_k^{\frac{1}{2}}(f))|^2 \right)^{\frac{1}{2}}.$$

By (12), applying Schwarz inequality, we have

$$\begin{aligned} |\sigma_N^0(f)| &\leqslant \sum_{k=0}^{N} A_{N-k}^{-\frac{1}{2}} A_k^{-\frac{1}{2}} |\sigma_k^{-\frac{1}{2}}(f) - \sigma_k^{\frac{1}{2}}(f)| + \sum_{k=0}^{N} A_{N-k}^{-\frac{1}{2}} A_k^{-\frac{1}{2}} |\sigma_k^{\frac{1}{2}}(f)| \\ &\leqslant \left(\sum_{k=0}^{N} (A_{N-k}^{-\frac{1}{2}})^2 \right)^{\frac{1}{2}} \delta(f) + \sigma_*^{\frac{1}{2}}(f) \\ &\leqslant C \log^{\frac{1}{2}}(N+2)\delta(f) + \sigma_*^{\frac{1}{2}}(f), \end{aligned} \tag{13}$$

where C is a proper constant.

Lemma 1. *The operator δ is bounded from $L^2_{\frac{1}{2}}(\Sigma_{n-1})$ to $L^2(\Sigma_{n-1})$; i.e.*

$$||\delta(f)|| \leqslant C||f_{\frac{1}{2}}|| \quad \text{for } f \in L^2_{\frac{1}{2}}(\Sigma_{n-1}),$$

where C is a proper constant.

Proof. We have

$$\begin{aligned} \sigma_k^{-\frac{1}{2}}(f) - \sigma_k^{\frac{1}{2}}(f) &= \frac{1}{A_k^{-\frac{1}{2}}} \sum_{j=0}^{k} A_{k-j}^{-\frac{1}{2}} \left(1 - \frac{A_k^{-\frac{1}{2}} A_{k-j}^{\frac{1}{2}}}{A_{k-j}^{-\frac{1}{2}} A_k^{\frac{1}{2}}} \right) Y_j(f) \\ &= \frac{1}{A_k^{-\frac{1}{2}}} \sum_{j=0}^{k} A_{k-j}^{-\frac{1}{2}} \frac{j}{k+\frac{1}{2}} Y_j(f). \end{aligned}$$

So,

$$\begin{aligned} ||\delta(f)||^2 &= \int_{\Sigma_{n-1}} \sum_{k=1}^{\infty} \left| \sum_{j=1}^{k} A_{k-j}^{-\frac{1}{2}} \frac{j}{k+\frac{1}{2}} Y_j(f)(x) \right|^2 dx \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{k} \left| A_{k-j}^{-\frac{1}{2}} \frac{j}{k+\frac{1}{2}} \right|^2 ||Y_j(f)||^2 \\ &\leqslant C \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \frac{j^2}{(k-j+1)k^2} ||Y_j(f)||^2 \end{aligned}$$

$$\leqslant C\sum_{j=1}^{\infty} \log(j+1)||Y_j(f)||^2$$
$$\leqslant C||f_{\frac{1}{2}}||^2. \qquad \square$$

Definition 3. Let $\alpha > -1$. For $f \in L^2(\Sigma_{n-1})$, define

$$\rho_\alpha(f)(x) := \sup\{\log^\alpha N|\sigma_N^0(f)(x)| : \ N \geqslant 3\} \tag{14}$$

$$h_\alpha(f)(x) := \sup\{\log^\alpha N|f(x) - \sigma_N^0(f)(x)| : \ N \geqslant 3\}. \tag{15}$$

Lemma 2. *For* $f \in L^2_{\frac{1}{2}}(\Sigma_{n-1})$,

$$||\rho_{-\frac{1}{2}}(f)|| \leqslant C||f_{\frac{1}{2}}||.$$

Proof. This is a direct consequence of (13), Lemma 1, and the boundedness of $\sigma_*^{\frac{1}{2}}$. $\square$

Corollary 1. *If* $f \in L^2_{\frac{1}{2}}(\Sigma_{n-1})$, *then*

$$\lim_{N\to\infty} \log^{-\frac{1}{2}} N|\sigma_N^0(f)(x)| = 0 \quad \textit{almost everywhere.}$$

Proof. By Definition 3, it is obvious that

$$h_{-\frac{1}{2}}(f)(x) \leqslant \rho_{-\frac{1}{2}}(f)(x) + |f(x)|.$$

Hence $||h_{-\frac{1}{2}}(f)|| \leqslant C||f_{\frac{1}{2}}||$ by Lemma 2. Given $\varepsilon > 0$, we choose $m \in \mathbb{N}$ big enough such that

$$||f - \sigma_m^0(f)|| \leqslant ||f_{\frac{1}{2}} - \sigma_m^0(f_{\frac{1}{2}})|| < \varepsilon$$

and write $g = \sigma_m^0(f)$ for simplicity. Since

$$\limsup_{N\to\infty} \log^{-\frac{1}{2}} N|\sigma_N^0(f)(x)| = \limsup_{N\to\infty} \log^{-\frac{1}{2}} N|\sigma_N^0(f-g)(x)| \leqslant h_{-\frac{1}{2}}(f-g)(x),$$

we get

$$\left|\left|\limsup_{N\to\infty} \log^{-\frac{1}{2}} N|\sigma_N^0(f)|\right|\right| \leqslant C||(f-g)_{\frac{1}{2}}|| < C\varepsilon.$$

So, by the arbitrariness of ε, the left-hand side of the above inequality is zero. $\square$

4. Proof of the theorems

We first prove the following lemma.

Lemma 3. *For all $s>0$ and $r>-1$, the condition $(\{s,r\})$ implies the condition $(\{2,r\})$.*

We will verify this by using K-functionals concerning the derivatives. Let $f \in L^2(\Sigma_{n-1})$ and $s>0$. If there exists a function $g \in L^2(\Sigma_{n-1})$ such that

$$g \sim \sum_{k=1}^{\infty} (k(k+n-2))^{\frac{s}{2}} Y_k(f),$$

then g is called the derivative of degree s of f and is written as $g = D^s f = f^{(s)}$.

Following [5], the sth K-functional $K_s(\cdot, t)$ on $L^2(\Sigma_{n-1})$ is defined by

$$K_s(f,t) = \inf\{||f-g|| + t^s ||g^{(s)}|| : g^{(s)} \in L^2(\Sigma_{n-1})\}.$$

Lemma 4 (see Ditzian [3]). *If $0<\alpha<\beta$, then*

$$K_\alpha(f,t) \leqslant C(\alpha,\beta) t^\alpha \int_t^1 K_\beta(f,u) u^{-\alpha-1}\, du.$$

Lemma 5 (see Kalyabin [5]). *Suppose $s>0$ and $f \in L^2(\Sigma_{n-1})$. Then*

$$\omega_s(f,t) \leqslant B_{n,s} K_s(f,t) \leqslant B'_{n,s} \omega_s(f,t) \quad \text{for all } t>0.$$

By Lemmas 2 and 3, we get directly that, for $0<\alpha<\beta<\infty$,

$$\omega_\alpha(f,t) \leqslant C(\alpha,\beta) t^\alpha \int_t^1 \frac{\omega_\beta(f,u)}{u^{\alpha+1}} du. \tag{16}$$

Proof of Lemma 3. Assume $0<s<2$. By (6) we know that if the condition $(\{s,r\})$ holds, then $(\{2,r\})$ holds also. Now we assume that $2<s$ and $(\{s,r\})$ holds. Then, by (16),

$$\omega_2(f,t) \leqslant C_s t^2 \int_t^1 \frac{\omega_s(f,u)}{u^3} du$$

and hence, by Schwarz inequality,

$$\omega_2(f,t)^2 \leqslant C_s t^4 \int_t^1 \frac{\omega_s(f,u)^2}{u^2} du \int_t^1 u^{-4}\, du \leqslant C_s t \int_t^1 \frac{\omega_s(f,u)^2}{u^2} du.$$

Then we get

$$\begin{aligned}\int_0^1 \frac{\omega_2(f,t)^2}{t}\log^r\left(\frac{2}{t}\right)dt &\leqslant C_s\int_0^1 \log^r\left(\frac{2}{t}\right)\int_t^1 \frac{\omega_s(f,u)^2}{u^2}\,du\,dt\\ &\leqslant C_s\int_0^1 \frac{\omega_s(f,u)^2}{u^2}\left(\int_0^u \log^r\left(\frac{2}{t}\right)dt\right)du\\ &\leqslant C_s\int_0^1 \frac{\omega_s(f,t)^2}{t}\log^r\left(\frac{2}{t}\right)dt. \qquad \square\end{aligned}$$

From Lemma 3, to prove Theorems 1 and 2, it suffices for us to prove both for $s=2$.

Proof of Theorem 1. Assume $r \geqslant 1$.

Let $f \in L^2(\Sigma_{n-1})$. If f satisfies the condition $(\{2,r\})$, then by Theorem 3,

$$f \in L^2_{\frac{r+1}{2}}(\Sigma_{n-1}) \subset L^2_1(\Sigma_{n-1}).$$

We first consider the case of $r=1$. In this case the result is known (see [2]). In fact, for any function $f \in L^2_1$, the almost everywhere convergence of the orthogonal expansion of f holds by a general theorem for orthogonal series (see [4, p. 190 for Russian translation]). But for the completeness we give a very short proof here. Generally, for $\alpha > 0$ and $f \in L^2_{\alpha+\frac{1}{2}}(\Sigma_{n-1})$, by using Abel transform twice, we have, for $N \geqslant 3$,

$$\begin{aligned}\sigma_N^0(f) &= \sum_{k=0}^{N} Y_k(f) = \sum_{k=0}^{N} \mu_k^\alpha Y_k(f_\alpha)\\ &= \sum_{k=0}^{N-2} (k+1)\Delta^2\mu_k^\alpha\sigma_k^1(f_\alpha) + (N-1)\Delta\mu_{N-1}^\alpha\sigma_{N-1}^1(f_\alpha) + \mu_N^\alpha\sigma_N^0(f_\alpha),\end{aligned}$$

where

$$\mu_k^\alpha := \frac{1}{\log^\alpha(k+2)},\ \ \Delta\mu_k^\alpha := \mu_k^\alpha - \mu_{k+1}^\alpha,\ \ \Delta^2\mu_k^\alpha := \Delta\mu_k^\alpha - \Delta\mu_{k+1}^\alpha.$$

Then we get

$$\sigma_*^0(f) \leqslant C_\alpha(\sigma_*^1(f_\alpha) + \rho_{-\alpha}(f_\alpha)).$$

Hence, by Lemma 2 and the boundedness of σ_*^1, we get

$$||\sigma_*^0(f)|| \leqslant C||\sigma_*^1(f_{\frac{1}{2}})|| + ||\rho_{-\frac{1}{2}}(f_{\frac{1}{2}})|| \leqslant C||f_1||.$$

Thus, for all $f \in L_1^2(\Sigma_{n-1})$,

$$\lim_{N\to\infty} \sigma_N^0(f)(x) = f(x) \quad \text{almost everywhere.} \tag{17}$$

Next we assume $r>1$. Fix $N>2$ temporarily and let $m>N$. Then

$$\sigma_m^0(f)-\sigma_{N-1}^0(f)=\sum_{k=N}^{m} Y_k(f)=\sum_{k=N}^{m}\mu_k^{\frac{r}{2}}Y_k(f_{\frac{r}{2}}).$$

Using Abel transform, we get

$$\sigma_m^0(f)-\sigma_{N-1}^0(f)=\sum_{k=N}^{m-1}\Delta\mu_k^{\frac{r}{2}}\sigma_k^0(f_{\frac{r}{2}})+\mu_m^{\frac{r}{2}}(\sigma_m^0(f_{\frac{r}{2}})-\sigma_{N-1}^0(f_{\frac{r}{2}})).$$

Since $f_{\frac{r}{2}}\in L^2_{\frac{1}{2}}(\Sigma_{n-1})$, by Corollary 1 we know

$$\lim_{m\to\infty}\mu_m^{\frac{r}{2}}(\sigma_m^0(f_{\frac{r}{2}})-\sigma_{N-1}^0(f_{\frac{r}{2}}))=0 \quad \text{almost everywhere } (r\geqslant 1). \tag{18}$$

Taking the limit $m\to\infty$ and applying (18) yield

$$f-\sigma_{N-1}^0(f)=\sum_{k=N}^{\infty}\Delta\mu_k^{\frac{r}{2}}\sigma_k^0(f_{\frac{r}{2}}) \quad \text{almost everywhere.}$$

Notice that $|\Delta\mu_k^{\frac{r}{2}}|\leqslant C\dfrac{1}{k\log^{1+\frac{r}{2}}k}$ $(k>2)$. We get

$$|f-\sigma_{N-1}^0(f)|\leqslant C\sum_{k=N}^{\infty}\frac{1}{k\log^{1+\frac{r-1}{2}}k}\rho_{-\frac{1}{2}}(f_{\frac{r}{2}})\leqslant\frac{C}{\log^{\frac{r-1}{2}}N}\rho_{-\frac{1}{2}}(f_{\frac{r}{2}}) \quad \text{almost everywhere.}$$

Applying Lemma 2, we obtain

$$||h_{\frac{r-1}{2}}(f)||\leqslant C||f_{\frac{r+1}{2}}||,$$

which implies, by a routine argument, the conclusion of Theorem 1 for $r>1$. □

Proof of Theorem 2. Assume $0\leqslant r<1$ and f satisfies $(\{2,r\})$. We write $\alpha=\frac{r-1}{2}$ for convenience. Then $-\frac{1}{2}\leqslant\alpha<0$. By Theorem 3, $f\in L^2_{\alpha+1}(\Sigma_{n-1})$. We have, for $N>3$,

$$\sigma_N^0(f)=\sum_{k=0}^{N-1}\Delta\mu_k^{\frac{r}{2}}\sigma_k^0(f_{\frac{r}{2}})+\mu_N^{\frac{r}{2}}\sigma_N^0(f_{\frac{r}{2}}).$$

Then

$$\begin{aligned}\log^{\alpha}(N+2)|\sigma_N^0(f)|&\leqslant C\sum_{k=0}^{N-1}\frac{\log^{\alpha}(N+2)}{(k+2)\log^{\alpha+\frac{3}{2}}(k+2)}|\sigma_k^0(f_{\frac{r}{2}})|+\frac{1}{\log^{\frac{1}{2}}(N+2)}|\sigma_N^0(f_{\frac{r}{2}})|\\&\leqslant C_{\alpha}\rho(f_{\alpha+1}).\end{aligned}$$

Therefore,

$$||h_{\alpha}(f)||\leqslant C_{\alpha}||f_{\alpha+1}||=C_{\alpha}||f_{\frac{r+1}{2}}||.$$

By this we finish the proof. □

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